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LETTER TO THE EDITOR

The generalised six-vertex model

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Abstract. In this letter we give conditions under which a certain, very general, version of the ice model of lattice statistics can be solved. Special cases of this model include the eight-vertex model, the three-colour problem and the staggered ice model soluble by the Pfaffian method. All of these special cases are shown to satisfy our criteria of solubility.

Various models of phase transitions defined on periodic lattices have received considerable theoretical attention. Of importance in this respect are the vertex models (for a review see Lieb and Wu 1972) and in particular the eight-vertex model (Fan and Wu 1970). This last problem was solved, in the symmetrical case, by Baxter (1971, 1972, 1973a, b, c). In the literature on the eight-vertex model, a so-called generalised ice-type model is encountered (see Kumar 1974, § 3(d)), which should not be confused with the general six-vertex model of Sutherland *et al* (1967).

In this letter, we shall consider this very general problem as a lattice model in its own right. We do not go into great detail but we shall present a set of conditions that the weights of the model must satisfy. Some special cases that meet these requirements are then considered.

Let us now define the model. Consider a square lattice of M rows and N columns and place an integer $m(I,J)$ on each face (I,J) (where $1 \leqslant I \leqslant M$ and $1 \leqslant J \leqslant N$) such that integers on adjacent faces differ by unity. We have therefore

$$|m(I,J)-m(I,J+1)| = |m(I+1,J)-m(I,J)| = 1 \tag{1}$$

together with the following boundary conditions

$$|m(M+1,J)-m(1,J)| = L\times\text{integer} \tag{2a}$$

and

$$|m(I,N+1)-m(I,1)| = L\times\text{integer} \tag{2b}$$

where L is some, as yet, unspecified integer and $m(M+1,J)$ and $m(I,N+1)$ are the values of the integers on the upper and right-hand borders of the lattice, respectively. One now defines a Boltzmann weight function for each group of four integers around each vertex (see figure 1),

$$W(I,J) \equiv W(m(I,J), m(I+1,J); m(I,J+1), m(I+1,J+1)).$$

$m(I+1, J)$	$m(I+1, J+1)$
$m(I, J)$	$m(I, J+1)$

Figure 1.

The reader may check that there are only six basically different types of weight function which we shall write as

$$\begin{aligned} &W(m, m-1; m+1, m) = a_m \qquad &&W(m, m+1; m-1, m) = a'_m \\ &W(m, m-1; m-1, m-2) = b_m \qquad &&W(m, m+1; m+1, m+2) = b'_m \\ &W(m, m-1; m-1, m) = c_m \qquad &&W(m, m+1; m+1, m) = c'_m. \end{aligned} \tag{3}$$

Figure 2 should make this clear.

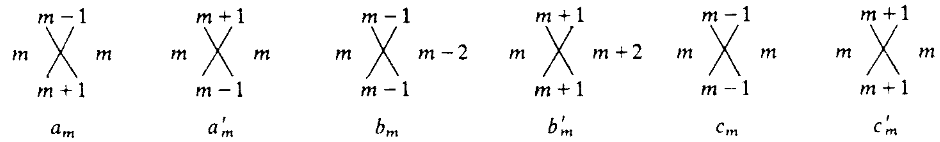

Figure 2.

An assumption we shall now make is that the weights are periodic in L, so that $a_{m+L} \equiv a_m, \ldots, c'_{m+L} \equiv c'_m$, for $m = 1, \ldots, L$. The partition function will be

$$Z_L = \sum \left\{ \prod_{I=1}^{M} \prod_{J=1}^{N} W(I, J) \right\} \tag{4}$$

where the summation is to be taken over all sets of integers which are distinct to modulus $L (L > 2)$ and the quantity in brackets is the product of weights for each set of integers $m(1, 1), m(1, 2), \ldots$, etc.

We shall now exhibit the conditions that the $6L$ weights $a_m, \ldots, c'_m$ must satisfy in order that the model be tractable by an extension of the Bethe ansatz (see Lieb 1967 for an introduction). We first define two intermediate parameters

$$\alpha_m = a_m a'_m - c_m c'_m \tag{5a}$$

and

$$\beta_m = b'_{m-1} b_{m+1} \tag{5b}$$

so that we can now introduce four further parameters

$$r_m = \alpha_m \beta_m \tag{6a}$$

$$s_m = -r_m c'_m c_{m+1} + c_m \beta_m c'_{m+1} \beta_{m+1} \tag{6b}$$

$$t_m = c'_{m-1} c_m + c'_m c_{m+1} + \alpha_m c_m^{\prime -1} c_{m+1} + c_{m-1} \beta_{m-1} c_m^{\prime -1} \tag{6c}$$

and

$$u_m = (\alpha_m + c_m c'_m) c'_{m-1} c_{m+1} - \beta_m c_m c'_m. \tag{6d}$$

The conditions are now most conveniently expressed in the following way,

$$r_m = r_{m+1} \tag{7a}$$

$$s_m = s_{m+1} \tag{7b}$$

$$t_m = t_{m+1} \tag{7c}$$

$$u_m = u_{m+1} \tag{7d}$$

for $m = 1, \ldots, L$. In other words, the functions of the weights (6*a*)–(6*d*) must not involve the integer m explicitly. Strictly speaking, (7*d*) is a consequence of the other three, but it cannot be written as a function of r_m, s_m and t_m. A full derivation of equations (6) is given in the author's thesis (see Pegg 1981).

We now wish to indicate that special, soluble cases of this model exist.

The eight-vertex model. To show that the symmetrical eight-vertex model (Baxter 1972) can be reduced to a special case of the generalised model requires a lengthy discussion. The weights (3) can be shown to be complicated functions of certain parameters which depend on m. One may show that, after lengthy algebraic manipulations, all m-dependent terms drop out of the invariants (6) which assume the following values

$$\begin{aligned} r_m &= (a^2-c^2)(b^2-d^2) \qquad & s_m &= (b^2c^2-a^2d^2)(-a^2+b^2+c^2-d^2) \\ t_m &= a^2+b^2+c^2+d^2 \qquad & u_m &= (a^2-b^2)(c^2-d^2) \end{aligned} \tag{8}$$

where a, b, c and d are the Boltzmann weights of the eight-vertex model. Thus the eight-vertex model has been shown to be soluble by algebraic means alone (Kumar 1974). There are certain restrictions on the integer L but these are of a rather technical nature (Baxter 1973a, § 1).

The three colouring model. This problem concerns the number of ways of colouring the faces of a square lattice using only three colours (1, 2 and 3 say) such that no two contiguous faces are coloured alike. One generalises the problem further (Baxter 1970) by giving the colours 'activities' z_1, z_2 and z_3. One has then to evaluate a partition function of the form

$$Z = \sum_{n_1}\sum_{n_2}\sum_{n_3} g(n_1, n_2, n_3) z_1^{n_1} z_2^{n_2} z_3^{n_3} \tag{9}$$

where $g(n_1, n_2, n_3)$ is the number of ways of colouring the lattice with n_1 faces coloured 1, n_2 coloured 2 and n_3 coloured 3. This problem is readily reduced to a generalised six-vertex model. We set $L = 3$ and

$$z_m = \xi_m^4 \, (\xi_{m+3} \equiv \xi_m) \tag{10}$$

for $m = 1, 2$ and 3. The weights (3) are easily seen to be

$$\begin{aligned} a_m &= a'_m = \xi_{m-1}\xi_m^2\xi_{m+1} \qquad & b_m &= \xi_{m-1}^2\xi_m\xi_{m+1} \\ c_m &= c'_{m-1} = \xi_{m-1}^2\xi_m^2 \qquad & b'_m &= \xi_{m-1}\xi_m\xi_{m+1}^2 \end{aligned} \tag{11}$$

and the invariants are easily worked out. They are

$$r_m = u_m = 0 \qquad s_m = (z_1 z_2 z_3)^2 \qquad t_m = z_1 z_2 + z_2 z_3 + z_3 z_1 \tag{12}$$

which are of course independent of m and the model is soluble. We note in passing that the values of (12) are those of an eight-vertex model with $a=b=c$ and with

$$z_1z_2z_3=a|a^2-d^2| \qquad z_1z_2+z_2z_3+z_3z_1=3a^2+d^2. \tag{13}$$

The staggered six-vertex model. Our final special case is the so-called staggered ice model of Wu and Lin (1975). We shall assume L to be an even integer and we shall set

$$\begin{gathered} a_m=a_{+,-} \qquad a'_m=a'_{+,-} \qquad b_m=b_{+,-} \qquad b'_m=b'_{+,-} \\ c_m=c_{+,-} \qquad c'_m=c'_{+,-} \end{gathered} \tag{14}$$

for m even and odd, respectively. One may easily convince oneself that, chessboard fashion, the even values of m alternate with those of odd m (M and N are assumed to be even integers). The conditions (7) now become quite simple and are

$$r_+=r_- \qquad s_+=s_- \qquad t_+=t_- \qquad u_+=u_- \tag{15}$$

in an obvious notation. They are found to be satisfied when

$$a_+a'_+ +b_+b'_+ -c_+c'_+ =0 \qquad a_-a'_- +b_-b'_- -c_-c'_- =0 \tag{16}$$

which are the free-fermion conditions of Wu and Lin who used the algebra of Pfaffians. The invariants (6) have the values

$$\begin{aligned} r_m&=-b_+b'_+b_-b'_- & s_m&=b_+b'_+b_-b'_-(c_+c'_- +c'_+c_-) \\ t_m&=c_+c'_- +c'_+c_- & u_m&=a_+a'_+a_-a'_- -b_+b'_+b_-b'_- \end{aligned} \tag{17}$$

for even and odd values of m.

In conclusion, then, we have shown that the generalised ice model can be shown to include two other models as special cases as well as the eight-vertex model. We have also given a set of conditions of solubility which must hold for the model to be, in principle, soluble. All of the special cases discussed in this letter satisfy these conditions but the solution of the general model is beyond the scope of this letter.

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